Spectral Methods for Fractional Advection-Dispersion Equations: Analytical and Numerical Perspectives

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Abstract:

In this paper we investigate analytical and numerical approaches for solving fractional diffusion and advection–diffusion equations involving the fractional Laplacian operator. Due to the inherent complexity of fractional-order models, obtaining closed-form solutions remains a challenging task. We use spectral definition of Fractional Laplacian $(-\Delta)\alpha/2$ on a bounded domain Ω. We use eigenfunctions to find analytical solution. To validate and compare the analytical approach, we implement a numerical method based on DST with IMEX Backward Euler scheme. We analyze the effect of the fractional order α on the solution profiles. Our results indicate that higher values of α approach classical diffusion behavior, while lower values exhibit heavy-tailed distributions characteristic of anomalous diffusion.

1. Introduction:

The advection–diffusion equation (ADE) serves as a fundamental mathematical tool for representing numerous physical systems across pure and applied sciences. Its significance is particularly evident in the study of transport phenomena, where it offers a natural and intuitive framework. In this context, advection refers to the bulk transport of matter caused by fluid motion, while diffusion captures the spontaneous tendency of particles to migrate from regions of higher concentration toward regions of lower concentration over time. This equation underlies the modeling of diverse real-world processes: movement of atmospheric pollutants such as smoke and dust, migration of contaminants in groundwater, dispersion of solutes in chemical solvents, intrusion of seawater into freshwater aquifers, and even thermal pollution in river systems. Because of these applications, the ADE has become a central subject of research across environmental science, heat and mass transfer, chemical engineering, and biological modelling.

Historically, the one-dimensional form of ADE has been widely studied. Classic examples include heat transfer in draining films [1], water movement in soils [2], saltwater intrusion into aquifers and pollutant transport in rivers [3], pollutant dispersion in shallow lakes [4], absorption of chemical species in solid beds [5], cooling of rigid materials by fluid flows [6], and thermal pollution in rivers [7]. Different numerical approaches have been employed to solve variations of the ADE. El-Baghdady and El-Azab applied the Legendre pseudospectral method to parabolic ADEs with variable coefficients and Dirichlet boundary conditions [8]. Prabhakaran and Doss developed a finite volume scheme for the one-dimensional case [9]. Closed-form approximations for nonlinear heat and mass diffusion equations with Dirichlet and Neumann conditions were investigated by Hristov [10]. Buske et al. [11] analyzed a coupled Navier–Stokes–ADE system to model pollutant concentration and wind-driven transport using the Adomian decomposition method. Grant and Wilkinson [12] studied a homogeneous ADE involving a time-dependent diffusion tensor, drift velocity, and absorbing boundaries.

More recently, interest has shifted toward fractional-order extensions of diffusion models. AlRefai [13] investigated both linear and nonlinear fractional differential equations and established conditions for solution existence. With the growing prominence of fractional calculus, extensive reviews [14–17] have highlighted its role in describing anomalous diffusion and memory effects. Two widely used operators, the Riemann–Liouville (R–L) and Caputo fractional derivatives, have been shown to capture realistic system dynamics [18–20]. However, each presents challenges: the R–L derivative yields physically inconsistent results such as a nonzero derivative for constants, while the Caputo derivative—although resolving these issues—relies on a singular kernel that complicates computation.

Within fractional modelling, along with Grunwald–Letnikov and Riemann–Liouville definitions, the fractional Laplacian $\Delta{-\alpha/2}$ where $\alpha \in (0,2)$ plays a central role due to its multiple equivalent representations [21]. Yet, difficulties arise when the problem is posed on bounded domains: in such settings, different mathematical interpretations must be invoked to handle boundary conditions [22,23]. The literature, however, has not yet reached agreement on which formulation of the fractional Laplacian is the most appropriate for bounded-domain applications [28-30]. In this work we use spectral approach of defining Fractional Laplacian.

This work adopts the Riesz fractional Laplacian approach, following the methodology of Li et al. [24], to model long-distance forward and backward tracer transport on a bounded domain . We solve the resulting Fractional Advection-Diffusion Equation (FADE) [25] using sine eigenfunction expansion and then we use spectral Discrete Sine Transform with IMEX (Implicit - Explicit) Backward Euler for numerical comparison.

This paper investigates analytical and numerical solutions for fractional diffusion equations governed by a fractional Laplacian operator. The structure of the paper is as follows: Section 2 establishes the mathematical foundation, beginning with the classical diffusion equation and extending it to the anomalous case via the fractional Laplacian using spectral definition. In Section 3, we give analytical solution to the Fractional Advection-Diffusion Equation (FADE) using, while section 4 deals with the numerical solution using Discrete sine transform and IMEX Backward Euler technique. We give conclusion in section 5.

2. Mathematical Formulation

2.1 The Classical Diffusion Equation

A linear diffusion equation over a computational domain Ω is given by [26]:

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t),\ x \in \Omega \quad (1)$$

where D is the diffusion coefficient and Δ represents the Laplacian operator, x is general spatial coordinate, t is time and $u(x,t)$ represents concentration at position x in time t. The boundary conditions vary as per the model of diffusion. In real situations the conditions could be more complex than the usual, such as Dirichlet, Neumann and mixed boundary conditions.

2.2 Anomalous Diffusion

For sufficiently smooth and decaying functions u(x,t), the fractional Laplacian is defined as an integral operator defined over the domain Rn . It is characterized by its definition as a pseudodifferential operator with the symbol |k|α as follows [27]:

$$(-\Delta)^{\alpha/2}u(x) = F^{-1}[|k|^{\alpha}F(u)],\ \alpha > 0. \quad (2)$$

where F represents the Fourier transform applied over the entire space Rn , and F−1 denotes Fourier inverse transform. When α = 2 , Equation (2) simplifies to the familiar spectral representation of the classical Laplace operator (−Δ).

The fractional Laplacian can be expressed using hypersingular integral as follows [27]:

$$(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+\alpha}} dy,\ \alpha \in (0,2). \quad (3)$$

where P.V. denotes the principal value integral, Cn,α is normalization constant, which is given by $\frac{2^{\alpha-1}\alpha\Gamma((\alpha+n)/2)}{\sqrt{\pi^n}\Gamma(1-\alpha/2)}$ and Γ(z) denotes the gamma function.

A key implication of Equation (3) is that evaluating the fractional Laplacian (−Δ)α/2u(x) at a point x ∈ Rn depends on the function u over the entire domain.

This global dependence poses a significant challenge for practical applications, where tracer transport must be modeled within bounded domains. Unlike its classical counterpart, the fractional Laplacian's theoretical framework and numerical treatment on bounded domains are less developed. A primary difficulty is reconciling its inherent nonlocality with finite computational domains. This nonlocality introduces complications such as broken translational invariance and long-range spatial correlations when boundary conditions are present [28]. One common strategy for defining the operator on a bounded domain Ω is to restrict the real-space definition in Formula (3) to Ω, resulting in the so-called Riesz fractional Laplacian [21]

$$(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha}\int_{\Omega}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}dy + \int_{\mathbb{R}^n\setminus\Omega}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}dy. \quad (4)$$

2.3 Definitions and Preliminaries

Definition 1 (The Schwartz space)**.** *The Schwartz space* S(R^n) *is the space of all smooth functions* $f : R^n \to C$ *such that* *f and all its derivatives decay faster than the reciprocal of any polynomial at infinity. i.e.* $\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \sup_{\mathbf{x}\in\mathbb{R}^n} |\mathbf{x}^\alpha \partial^\beta f(\mathbf{x})| < \infty \quad \forall \alpha, \beta \in \mathbb{N}_0^n \right\}$

.

Definition 2 (Fourier Transform)**.** *The Fourier transform of a function* $f(\mathbf{x}) \in S(R^n)$ *is defined as:*

$$F\{f(\mathbf{x})\} = \hat{f}(\xi) = \int_{R^n} f(\mathbf{x})e^{-2\pi i\mathbf{x}\cdot\xi}\, d\mathbf{x}. \quad (5)$$

Definition 3 (Inverse Fourier Transform)**.** *The inverse Fourier transform of a function* $\hat{f}(\xi) \in S(R^n)$ *is defined as:* Z

$$F^{-1}\{\hat{f}(\xi)\} = f(x) = \int_{R^n} \hat{f}(\xi)e^{2\pi i\mathbf{x}\cdot\xi}\, d\xi. \quad (6)$$

Definition 4. Laplacian (Spectral Definition). Let **Ω** = (0,L) and consider the Dirichlet Laplacian operator −Δ on **Ω**. The eigenvalue problem is given by

$$-\Delta\phi_n(x) = \lambda_n^2\,\phi_n(x), \qquad x \in (0,L), \quad \phi_n(0) = \phi_n(L) = 0. \quad (7)$$

The normalized eigenfunctions are

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, \ldots \quad (8)$$

with eigenvalues

$$\lambda_n = \frac{n\pi}{L}. \quad (9)$$

The fractional Laplacian $(-\Delta)^{\alpha/2}$ with $1 < \alpha \leq 2$ is then defined spectrally as

$$(-\Delta)^{\alpha/2}\phi_n(x) = \lambda_n^\alpha\,\phi_n(x). \quad (10)$$

Definition 5. Sine Expansion. Any function $f \in L^2(0,L)$ satisfying homogeneous Dirichlet boundary conditions can be expanded in terms of the sine eigenfunctions:

$$f(x) = \sum_{n=1}^{\infty} f_n\,\phi_n(x), \qquad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad (11)$$

with Fourier sine coefficients

$$f_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \quad (12)$$

Remark 1. *Orthogonality. The sine eigenfunctions form an orthogonal basis in* $L^2(0,L)$*, satisfying*

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n, \\ \frac{L}{2}, & m = n. \end{cases} \tag{13}$$

Definition 6. *Inner Product. The inner product on* $L^2(0,L)$ *is defined by*

$$\langle f, g \rangle = \int_0^L f(x)g(x)\,dx. \tag{14}$$

3. Analytic Solution on a Finite Interval with Dirichlet Boundary Conditions

We consider the space-fractional advection–diffusion equation

$$R\frac{\partial C(x,t)}{\partial t} = -v\frac{\partial C(x,t)}{\partial x} - D(-\Delta)^{\alpha/2}C(x,t), \quad x \in (0,L),\ t > 0, \tag{15}$$

with initial condition

$$C(x,0) = C_0(x),\ x \in (0,L), \tag{16}$$

and Dirichlet boundary conditions

$$C(0,t) = C(L,t) = 0, \qquad t > 0, \tag{17}$$

where $R > 0$, $v \in \mathrm{R}$, $D > 0$, and $1 < \alpha \leq 2$.

Theorem 1. The fractional advection difusion equation given in (15) with initial condition (16) and boundary condition (17) satisfies

$$C(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L C_0(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right]\exp\left[-\frac{1}{R}\left(v\frac{n\pi}{L} + D\left(\frac{n\pi}{L}\right)^{\alpha}\right)t\right]\sin\left(\frac{n\pi x}{L}\right). \tag{18}$$

Proof. For the finite interval, we expand C(x,t) in the sine eigenfunctions of the Laplacian operator with Dirichlet boundary conditions:

$$C(x,t) = \sum_{n=1}^{\infty} c_n(t)\,\phi_n(x), \qquad \phi_n(x) = \sin\left(\frac{n\pi x}{L}\right). \tag{19}$$

The eigenfunctions satisfy

$$-\Delta\phi_n(x) = \left(\frac{n\pi}{L}\right)^2\phi_n(x),$$

so that the fractional Laplacian acts as

$$(-\Delta)^{\alpha/2}\phi_n(x) = \left(\frac{n\pi}{L}\right)^{\alpha}\phi_n(x).$$

Substituting into the PDE and projecting onto φ_n, we obtain for each mode:

$$R\frac{dc_n(t)}{dt} = -v\,\lambda_n\,c_n(t) - D\,\mu_n\,c_n(t), \tag{20}$$

where

$$\lambda_n = \frac{n\pi}{L}, \qquad \mu_n = \left(\frac{n\pi}{L}\right)^{\alpha}.$$

Thus, each coefficient satisfies an ODE:

$$\frac{dc_n(t)}{dt} = -\frac{1}{R}(v\lambda_n + D\mu_n)c_n(t). \tag{21}$$

$$c_n(t) = c_n(0)\exp\left[-\frac{1}{R}(v\lambda_n + D\mu_n)t\right] \tag{22}$$

where the initial modal coefficient is obtained from the sine expansion:

$$c_n(0) = \frac{2}{L}\int_0^L C_0(x)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{23}$$

Hence, the solution is given by the sine series:

$$C(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L C_0(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right]\exp\left[-\frac{1}{R}\left(v\tfrac{n\pi}{L} + D\left(\tfrac{n\pi}{L}\right)^{\alpha}\right)t\right]\sin\left(\frac{n\pi x}{L}\right). \tag{24}$$

Remark 2. Classical Advection-Diffusion ($\alpha = 2$) When $\alpha = 2$, the operator $(-\Delta)^{\alpha/2}$ reduces to the standard Laplacian, and the modal decay rate simplifies to

$$\frac{1}{R}(v\lambda_n + D\mu_n) = \frac{1}{R}\left(v\frac{n\pi}{L} + D\left(\frac{n\pi}{L}\right)^2\right).$$

Thus, the solution becomes

$$C(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L C_0(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right]\exp\left[-\frac{1}{R}\left(v\tfrac{n\pi}{L} + D\left(\tfrac{n\pi}{L}\right)^{2}\right)t\right]\sin\left(\frac{n\pi x}{L}\right). \tag{25}$$

This recovers the standard Fourier sine series solution of the classical advection-diffusion equation with homogeneous Dirichlet boundary conditions.

4. Spectral (FFT/DST) solution with IMEX Backward– Euler

We solve the initial-boundary value problem given by (15),(16) and (17) using spectral DST with IMEX Backward Euler.

4.1 Grid and discrete transforms

We choose a uniform interior grid with N interior points

$$x_j = jh, \qquad j = 1, \ldots, N, \qquad h = \frac{L}{N+1}.$$

and use the discrete sine transform pair consistent with the sine-eigenbasis for Dirichlet boundary conditions.

$$\widehat{u}_k = \frac{2}{N+1}\sum_{j=1}^{N} u_j \sin\Big(\frac{k\pi j}{N+1}\Big), \qquad k = 1,\ldots,N, \tag{26}$$

$$u_j = \sum_{k=1}^{N} \widehat{u}_k \sin\Big(\frac{k\pi j}{N+1}\Big), \qquad j = 1,\ldots,N. \tag{27}$$

With this convention the discrete sine modes correspond to the continuous eigenfunctions $\sin(k\pi x/L)$ evaluated at x_j because $\sin\left(\frac{k\pi x_j}{L}\right) = \sin\left(\frac{k\pi j}{N+1}\right)$.

The spectral derivative is computed from the sine coefficients:

$$\partial_x u(x_j) \approx \sum_{k=1}^{N} \widehat{u}_k \left(\frac{k\pi}{L}\right) \cos\Big(\frac{k\pi j}{N+1}\Big). \tag{28}$$

The spectral action of the spectral fractional Laplacian is diagonal in the sine basis:

$$(-\Delta)^{\alpha/2}\left(\sin\frac{k\pi x}{L}\right) = \left(\frac{k\pi}{L}\right)^{\alpha} \sin\frac{k\pi x}{L},$$

so in coefficient space we have the multipliers

$$\Lambda_k := \left(\frac{k\pi}{L}\right)^{\alpha}, \qquad k = 1,\ldots,N.$$

4.2 IMEX Backward–Euler time discretization

Let $u^n_j \approx u(x_j,t^n)$ denote the solution at time $t^n = n\Delta t$. We discretize in time using an IMEX Backward–Euler scheme: treat fractional diffusion implicitly and advection explicitly:

$$R\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v\big(\partial_x u\big)_j^n - D\left((-\Delta)^{\alpha/2}u\right)_j^{n+1}.$$

Project (i.e. take the discrete sine transform (26)) of both sides. Using linearity and the diagonal action of the fractional Laplacian in the sine basis, we obtain (for each mode k):

$$\frac{R}{\Delta t}\left(\widehat{u}_k^{n+1} - \widehat{u}_k^n\right) = -v\,\widehat{(\partial_x u)}_k^{\,n} - D\,\Lambda_k\,\widehat{u}_k^{n+1}. \tag{29}$$

Rearranging gives the explicit formula used in the algorithm:

$$\widehat{u}_k^{n+1} = \frac{\dfrac{R}{\Delta t}\widehat{u}_k^n - v\,\widehat{(\partial_x u)}_k^{\,n}}{\dfrac{R}{\Delta t} + D\,\Lambda_k}. \tag{30}$$

Equivalently we can write

$$\widehat{u}_k^{n+1} = \frac{R\,\widehat{u}_k^n - \Delta t\,v\,\widehat{(\partial_x u)}_k^{\,n}}{R + \Delta t\,D\,\Lambda_k}. \tag{31}$$

4.3 FFT/DST algorithm

1. **Set parameters.** Choose N (number of interior points), time step Δt, final time T, grid $x_j = jL/(N+1)$ for $j = 1,\dots,N$. Precompute $\Lambda_k = (k\pi/L)^\alpha$ for $k = 1,\dots,N$.

2. **Initial condition.** Sample $u^0_j = u_0(x_j)$ for $j = 1,\dots,N$.

3. **Time loop:** for $n = 0,1,\dots$ until $t^{n+1} = t^n + \Delta t > T$ do:

   a. *Forward DST.* Compute sine coefficients of the current solution:

$$\widehat{u}_k^n = \frac{2}{N+1}\sum_{j=1}^{N} u_j^n \sin\Big(\frac{k\pi j}{N+1}\Big), \qquad k = 1,\dots,N,$$

   using an FFT-based DST routine.

   b. *Spectral differentiation.* Compute the spatial derivative at grid points using the sine coefficients:

$$(\partial_x u)_j^n = \sum_{k=1}^{N} \widehat{u}_k^n \left(\frac{k\pi}{L}\right) \cos\Big(\frac{k\pi j}{N+1}\Big), \qquad j = 1,\dots,N.$$

   Evaluate this cosine series efficiently using a DCT (or via FFT-based routines that implement the required cosine transform).

   c. *Form right-hand side (physical space).* Define

$$r_j = R\,u_j^n - \Delta t\, v\,(\partial_x u)_j^n, \qquad j = 1,\dots,N.$$

   d. *DST of RHS.* Compute the sine coefficients of r:

$$\widehat{r}_k = \frac{2}{N+1}\sum_{j=1}^{N} r_j \sin\Big(\frac{k\pi j}{N+1}\Big) \quad .$$

   (Note: $\widehat{r}_k = R\widehat{u}_k^n - \Delta t\, v\,\widehat{(\partial_x u)}_k^{\,n}$.)

   e. Modal update (implicit diffusion). For each k = 1,...,N compute

$$\widehat{u}_k^{n+1} = \frac{\widehat{r}_k}{R + \Delta t\, D\,\Lambda_k}.$$

   f. Inverse DST. Recover the updated physical values using an inverse DST (FFT-based).

$$u_j^{n+1} = \sum_{k=1}^{N} \widehat{u}_k^{n+1} \sin\Big(\frac{k\pi j}{N+1}\Big), \qquad j = 1,\dots,N,$$

4. End time loop.

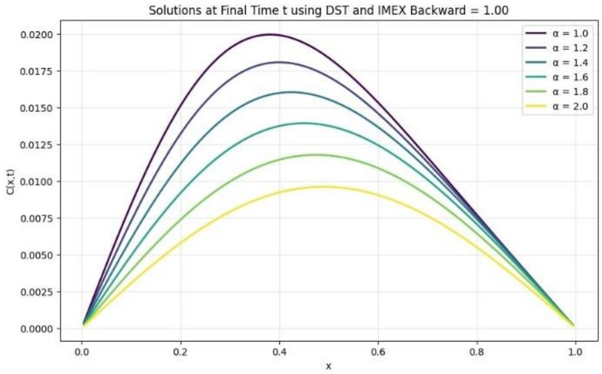

Figure 1: Using R = 1,v = 0.5 and D = 0.1 and Gaussian Initial Condition

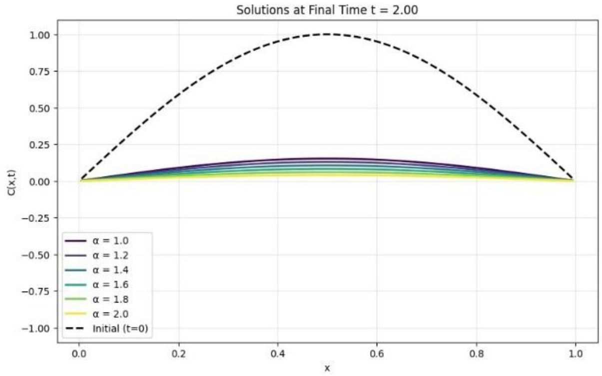

Figure 2: Using R = 1.v = 0.2 and D = 0.1 and $C_0 = \sin(\frac{n\pi x}{L})$

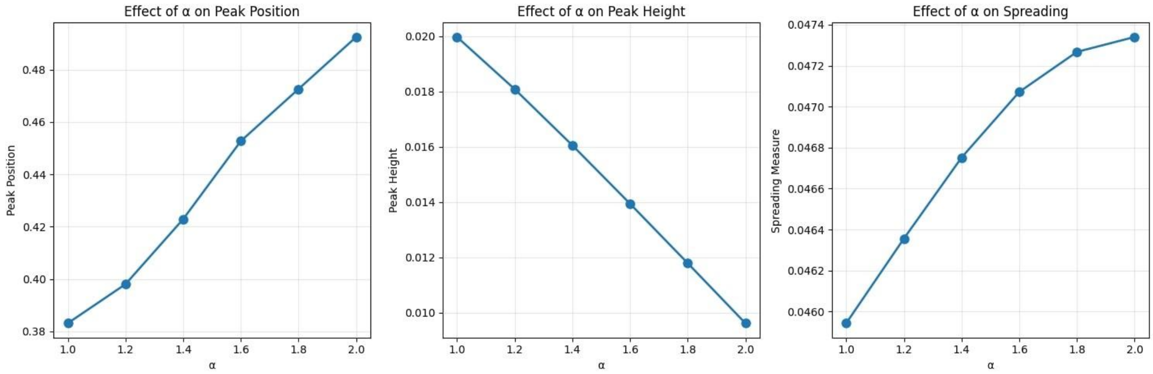

Figure 3: Comparison of various α's

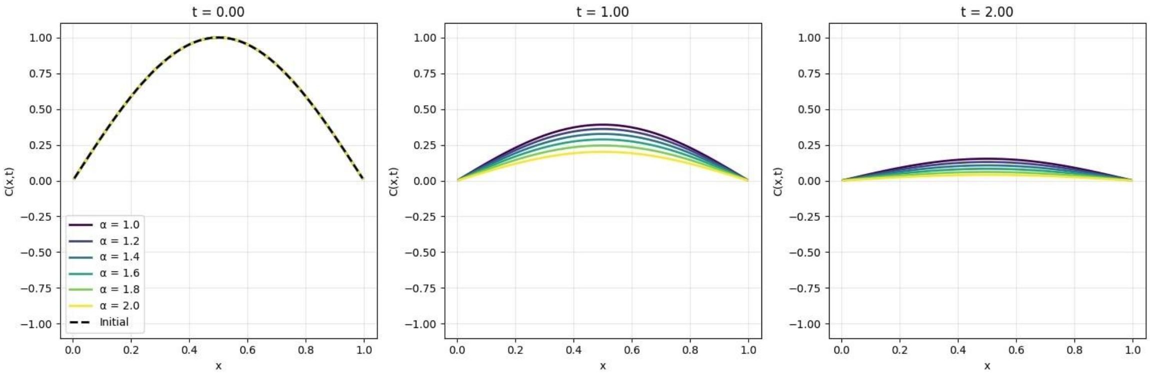

Figure 4: Comparison at various time

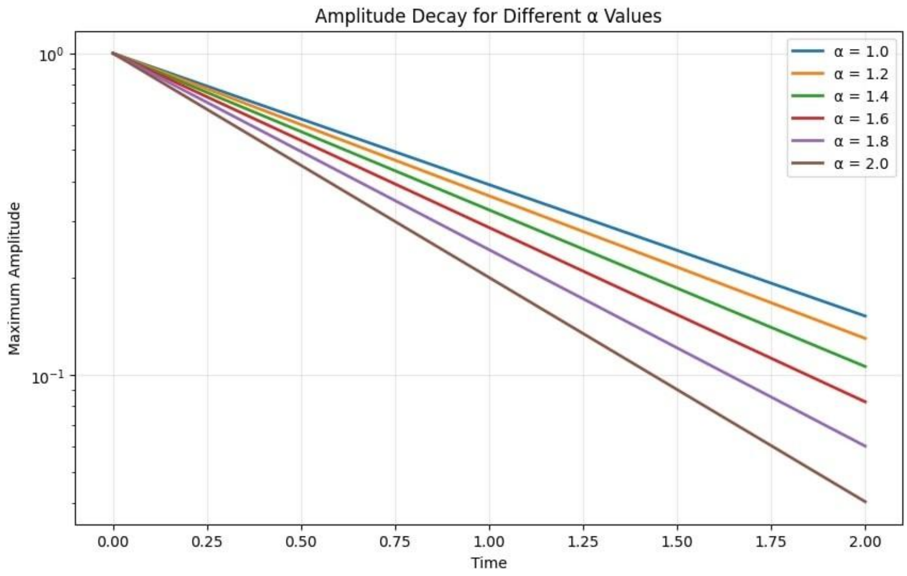

Figure 5: Ampitude Decay

5. Conclusion

Analytical solutions to fractional diffusion equations, whether linear or nonlinear, pose significant challenges due to their inherent nonlocality and memory effects. While classical integral transforms such as the Fourier, Laplace, and Fast Fourier transforms are commonly employed in solving these equations. We gave analytical solution to the fractional advection-diffusion equation using spectral approach. The behaviour observed is consistent: as the fractional order α increases toward 2, the system exhibits characteristics of normal diffusion, whereas smaller values of α lead to pronounced heavy-tailed distributions, reflecting the presence of anomalous diffusion.

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